Galois theory of Lie-Vessiot Systems

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1st Gifu Non-Linear WS
June, 2008

Outline

I. Settings

I.i. Motivation

- A superposition law for a non autonomous differential equation

$$\frac{dx_i}{dt} = F_i(t, x) \quad i = 1, \ldots, n$$

is a set of formulae,

$$\varphi_i(x^{(1)}, \ldots, x^{(r)}, \lambda),$$

expressing the general solution,

$$x(t) = \varphi(x^{(1)}(t), \ldots, x^{(r)}, \lambda)$$

as function of a fundamental system of solutions, and n arbitrary constants λ_i.

I.i. Motivation (II)

- The main example is the linear superposition for solutions of a linear system,

$$\dot{x} = A(t).x$$

- n linearly independent solutions give the general solution by linear combinations.
- This is the linear superposition law for linear equations,

$$\varphi\colon \mathbb{C}^n \times \mathbb{C}^n_\lambda \to \mathbb{C}^n$$

$$\varphi(x^{(1)}, \ldots, x^{(n)}, \lambda) = \sum_{i=1}^{n} \lambda_i x^{(i)}$$

I.i. Motivation (III)

- There are non-linear equations related to linear system. They also admit superposition Laws. The main example is the Riccati equation.

$$\dot{x} = a(t) + b(t)x + c(t)x^2$$

- The anharmonic ratio of 4 different solutions is a constant of the equation,

$$\frac{d}{dt}\frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_4)(x_2 - x_4)} = 0.$$

- We can then express the fourth solution as a function of three know solutions and the constant anharmonic ratio λ,

$$x = \frac{x_3(x_1 - x_2) - \lambda x_1(x_3 - x_2)}{(x_1 - x_2) - \lambda(x_3 - x_2)}.$$

I.i. Motivation (IV)

- We can find also genuine non-linear superposition laws.
- There is also a main example, the generalized Weierstrass equation:

$$\dot{x}^2 = f(t)(x^3 - g_2 x - g_3)$$

- Its general solution is of the form,

$$x(t) = \wp\left(\int_0^t f(\tau)d\tau + \lambda\right).$$

- Weierstrass' addition formula for $\wp$ function,

$$\wp(t+u) = -\wp(t) - \wp(u) + \frac{1}{4}\left(\frac{\wp'(t) - \wp'(u)}{\wp(t) - \wp(u)}\right)^2$$

is then a superposition formula for the solutions.

I.ii. Historical Develpement

- Differential equations admitting superposition laws were introduced By S. Lie in 1885.
- Local conditions for the existence of a superposition law were given by S. Lie and G. Scheffers in 1893. Some other advances in the earlier theory came from E. Vessiot and A. Gulbdberg.
- In the context of mathematical physics several authors carried out the research in 20th and 21th century: Winternitz, Shnider, Shorine, Cariñena, Grabowsky, Marmo, Ramos.
- Superposition laws are also interesting for differential algebraists. The work of K. Nishioka relates superposition Laws with Kolchin's strongly normal extensions of differential fields.
- Contemporary approach to SNE, due to J. Kovacic and R. Churchill is also related with superposition laws.

I.iii. Notation and Conventions

- We will set out the theory in the context of complex analytic geometry.
- The phase space is assumed to be a complex analytic manifold M. Denote $\mathcal{O}_M$ its sheaf of regular functions.
- The independent variable (t) moves on a Riemann surface S.
- We assume that S is endowed with a meromorphic derivation ∂. We denote by $S^\times$ to $S \setminus \{\text{poles of } \partial\}$.
- By a non-autonomous vector field $\vec{X}$ in M we mean an autonomous vector field in $M \times S^\times$ projectable onto ∂ in $S^\times$.

$$\vec{X} = \partial + \sum_{i=1}^{n} f_i(x,t)\frac{\partial}{\partial x_i}$$

- Local solutions of $\vec{X}$ are local sections of the fiber bundle $M \times S^\times \to S^\times$.

II. Lie's Superposition Theorem

II.i. Superposition Laws

- Let $\vec{X}$ be a non-autonomous vector field in M with time varying in S.
- For any integer, we consider the cartesian power M^r,

$$M^r = M^{(1)} \times \ldots \times M^{(r)}.$$

- We denote $\vec{X}^r$ to the vector field $\vec{X}$ lifted to the cartesian power M^r.

$$\vec{X}^r = \partial + \sum_{i,k=0}^{n,r} f_i(x^{(k)}, t) \frac{\partial}{\partial x_i^{(k)}}$$

- We say that an open subset $U \subset M^r$ is analytic if its complementary is a closed analytic subset of positive codimension.

II.i. Superposition Laws (II)

Definition

A superposition law for $\vec{X}$ is an analytic map,

$$\varphi\colon U \times M \to M,$$

where:

- U is an analytic susbet of M^r
- U is union of orbits of $\vec{X}^r$.

Such that, for any local solution $(x^{(1)}(t), \ldots, x^{(r)}(t))$ of $\vec{X}^r$ taking values in U, defined for t varying in $S' \subset S$,

$$x_\lambda(t) = \varphi(x^{(1)}(t), \ldots, x^{(r)}(t), \lambda)$$

is the general solution of $\vec{X}$ for t varying in S'.

II.ii. Lie-Vessiot-Guldberg Algebra

- Consider $\vec{X}$ a non-autonomous vector field in M with t varying in S.
- Denote $\mathfrak{X}(M)$ the Lie algebra of autonomous analytic vector fields in M.
- For $t_0 \in S^\times$, we define $\vec{X}_{t_0}$ in $\mathfrak{X}(M)$,

$$\vec{X}_{t_0} = \sum f_i(x, t_0)\frac{\partial}{\partial x_i}.$$

- $\vec{X}$ is seen as a map $\vec{X} \colon S^\times \to \mathfrak{X}(M)$

Definition

We call Lie-Vessiot-Guldberg algebra of $\vec{X}$ to the Lie algebra $\mathfrak{g}(\vec{X})$ of vector fields in M spanned by the image of the above defined map.

II.iii. Lie's Superposition Theorem (Local)

Theorem (Lie-Scheffers 1893)

A neccesary condition for the existence of a superposition law for $\vec{X}$ is $\dim \mathfrak{g}(X) < \infty$.

Definition

A local superposition law of $\vec{X}$ fo rank r is a foliation $\mathcal{F}$ of M^{r+1} verifiying:

- $\vec{X}^{r+1}$ is tangent to $\mathcal{F}$.
- $\mathcal{F}$ is of codimension greater or equal than n.
- $\mathcal{F}$ intersect transversally the fibers of the last projection $M^{r+1} \to M$.

Theorem (Cariñena, Grabowsky & Marmo 2006)

$\vec{X}$ admits a local superposition law if and only if $\dim \mathfrak{g}(\vec{X}) < \infty$.

II.iv. Complex Analytic Lie Groups

- Let G be a complex analytic Lie group.
- Denote by $\mathcal{R}(G)$ and $\mathcal{L}(G)$ the Lie algebras of right and left invariant vector fields respectively.
- Each one of these characterize the other by their commutativity:
$$[\mathcal{L}(G), \mathcal{R}(G)] = 0.$$
- We have the exponential map,
$$\mathbb{C} \times \mathcal{R}(G) \xrightarrow{\exp} G, \quad \mathbb{C} \times \mathcal{L}(G) \xrightarrow{\exp} G$$
- The flow of $A \in \mathcal{R}(G)$ is $\phi_A(t, \sigma) = e^{tA} \cdot \sigma$, and of $B \in \mathcal{L}(G)$, $\phi_B(t, \sigma) = \sigma \cdot e^{tB}$.

II.v. Pretransitive Lie Group Actions

- Consider an analytic action of G in M,

$$G \times M \to M, \quad (\sigma, x) \mapsto \sigma \cdot x.$$

- Right invariant vector fields project onto M. We obtain fundamental vector fields,

$$A \mapsto A^M, \quad \phi_{A^M}(t, x) = e^{tA} \cdot x.$$

- Denote $\mathcal{R}(G, M)$ the algebra of fundamental vector field,

$$\mathcal{R}(G) \to \mathcal{R}(G, M) \subset \mathfrak{X}(M),$$

- If the action is faithful, then this morphism is injective.

II.v. Pretransitive Lie Group Actions (II)

- Let G act faithfully in M.
- We lift tha action to each cartesian power M^r, component by component.
- A point $\bar{x} \in M^r$ is called *principal* is its orbit $G \cdot x$ is isomorphic to G (principal homogeneous space).

Definition

We say that the action of G in M is pretransitive if there are r and an invariant analytic open subset $U \subset M^r$ such that:

- U is made by principal points.
- The space of orbits U/G is an analytic manifold.

Theorem

If M is an homogeneous space of finite rank, then the action of G in M is pretransitive.

II.vi. Lie's Superposition Theorem (Global)

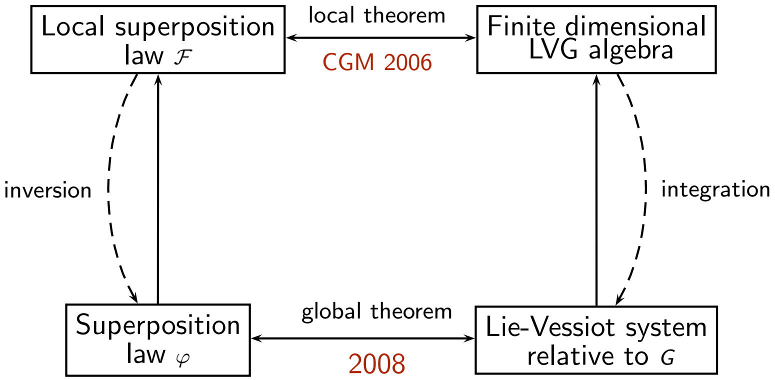
Definition

A non-autonomous vector field $\vec{X}$ is called a Lie-Vessiot system associated to the action of G in M if $\mathfrak{g}(\vec{X}) \subset \mathcal{R}(G, M)$.

Theorem (2008)

A non-autonomous vector field vec $\vec{X}$ in M admit a superposition law if and only if it is a Lie-Vessiot system associated to a pretransitive action in M.

II.vii. Lie's Superposition Theorem (Scheme)



III. Lie-Vessiot Hierarchy

III.i. Automorphic System (I)

Definition

We call automorphic system to any Lie-Vessiot system in a Lie group G.

- The superposition law for an automorphic system is the composition law,

$$G \times G \to G.$$

- For small $S' \subset S$ the space of solutions $\mathcal{G}_{\vec{A}}$ is a principal homogeneous space by the right side,

$$\mathcal{G}_{\vec{A}} \times G \to G, \quad (\sigma(t), \tau) \mapsto \sigma(t) \cdot \tau$$

- Example: Let $U(t)$ be a fundamental matrix of solutions of $\dot{x} = A(t).x$. For each non denerate constant matrix $C \in GL(n, \mathbb{C})$, $U(t).C = V(t)$ is another fundamental matrix of solutions.

III.i. Automorphic System (II)

Let M be a faithful G-homogeneous space of finite rank r.

- Let $\vec{X}$ be a Lie-Vessiot system in M with t varying in S. Then, there are functions $f_i(t)$ in $S^\times$ with:

$$\vec{X} = \sum_{i=1}^{s} f_i(t) A_i^M, \quad R(G, M) = \langle A_1^M, \ldots A_s^M \rangle.$$

- By the isomorphism $\mathcal{R}(G, M) \simeq \mathcal{R}(G)$ we define the associated automorphic system,

$$\vec{A} = \sum_{i=1}^{s} f_i(t) A_i.$$

- Example: $\dot{x} = A(t)x \rightsquigarrow \dot{U} = A(t)U$.

III.ii. Lie-Vessiot Hierarchy

- G-space morphisms,

$$\phi\colon M \to N, \quad \phi(\sigma \cdot x) = \sigma \cdot \phi(x),$$

  carry Lie-Vessiot systems onto Lie-Vessiot systems.
- Let $\vec{A}$ be an automorphic system in G. Throughout the morphism,

$$\mathcal{R}(G) \to \mathcal{R}(G, M),$$

  it induces a Lie-Vessiot system $\vec{A}^M$ in any G-space M.
- If M is a faithful G-homogeneous G-space, then $\vec{A}$ is the automorphic system associated to $\vec{A}^M$.
- Any G-space morphism $M \to N$ carry $\vec{A}^M$ onto $\vec{A}^N$.

III.iii. Representation of Solutions

- $\vec{A}$ automorphic system, $\vec{A}^M$ induced Lie-Vessiot in homogeneous G-space.
- For a solution $\sigma(t)$ of $\vec{A}$,

$$x_\lambda(t) = \sigma(t) \cdot \lambda,$$

  is the general solution of $\vec{A}^M$ for λ varying in M.
- If $M_1, \ldots, M_l$ are G-homogeneous spaces, such that the generic point of $\prod_{i=1}^{l} M_i$ is principal, then we are maps,

$$\varphi\colon M_1 \times \ldots M_l \times M \dashrightarrow M,$$

  such that, for $x^{(i)}$ particular solution of the $\vec{A}^{M_i}$,

$$x_\lambda(t) = \varphi(x^{(1)}(t), \ldots, x^{(l)}(t), \lambda),$$

  is the general solution of $\vec{A}^M$.

III.iii. Representation of Solutions (II)

- Superposition laws are the particular case $M_i = M$.
- D'Alambert reduction, integration or Riccati equation with a known solution, are examples of Representation formulas.

IV. Lie's Reduction Method

IV.i. Logarithmic Derivative

- Let us consider $S' \subset S^\times$.
- Let $\sigma(t) \in \mathcal{O}(S', G)$ be a S'-parameterized curve in G.
- For $t_0 \in S$ denote by $\sigma'(t_0)$ the tangent vector aot t_0.

$$\sigma' \colon T_{t_0}S' \to T_{\sigma(t_0)}S, \quad \partial_{t_0} \mapsto \sigma'(t_0).$$

- There is a unique $A(t_0) \in \mathcal{R}(G)$ such that $A_{\sigma(t_0)}$ is $\sigma'(t_0)$.
- The curve $S' \to \mathcal{R}(G)$, $t \mapsto A(t)$ is called logarithmic derivative of $\sigma(t)$.
- If identifying $\mathcal{R}(G)$ with $T_e G$,

$$l\partial(\sigma(t)) = R'_{\sigma(t)^{-1}}(\sigma'(t)).$$

IV.ii. Gauge Transformations

- Automorphic system $\vec{A}$ is equvalent to the following automorphic equation:

$$l\partial(x) = \vec{A} - \partial \quad (:= \vec{A}^{trans}).$$

- Logarithmic derivate of a composition is done as follows:

$$l\partial(\sigma(t)\tau(t)) = l\partial(\sigma(t)) + Adj_{\sigma(t)}(l\partial(\tau(t))).$$

- We want to take advantage of this, for frame changes in the automorphic equation.

IV.ii. Gauge Transformations (II)

- Consider $\sigma(t) \in \mathcal{O}(S', G)$. We define the map,

$$L_{\sigma(t)} \colon G \times S' \to G \times S', \quad (\tau, t) \mapsto (\sigma(t) \cdot \tau, t).$$

- It maps the automorphic system $\vec{A}$ to a new automorphic system.
- The new attached automorphic equation is,

$$l\partial(x) = Adj_{\sigma(t)}(\vec{A}^{trans}) + l\partial(\sigma(t)).$$

IV.iii. Lie's theorem on Reduction (local)

For $H \subset G$, we have $\mathcal{R}(H) \subset \mathcal{R}(G)$. Automorphic systems in H are, in particular, automorphic systems in G. Let us consider:

- $\vec{A}$ automorphic in G.
- M homogeneous G-space.
- $\vec{X}$ induced Lie-Vessiot in M.
- $x_0 \in M$, and $H \subset M$ its isotropy group.

Lemma

If $x(t) = x_0$ is a solution of $\vec{X}$, then $\vec{A}$ is an automorphic system in $H \subset G$.

IV.iii. Lie's theorem on Reduction (II)

- Let $x(t)$ be a particular solution of $\vec{X}$.
- Assume that there exist a curve $\sigma(t)$, such that $\sigma(t) \cdot x_0 = x(t)$.
- The Gauge transformantion $L_{\sigma(t)^{-1}}$ sends $\vec{A}$ to a new automorphic system $\vec{B}$.
- The lemma applies to $\vec{B}$.

The problem is to find a Global $\sigma(t)$!

IV.iii. Lie's theorem on Reduction (III)

For a given meromorphic $x\colon S \to M$ we consider.

$$\mathcal{H} = \{(\sigma, t) \,|\, \sigma \cdot x_0 = x(t)\} \subset G \times S,$$

projection $\pi\colon \mathcal{H} \to S$ is a fiber bundle. Our gauge transformation comes from a global section.

Theorem

Assume that there is a meromorphic solution $x(t)$ of $\vec{X}$ defined in S. Assume one of the following:

1. H is special
2. S is non-compact and H connected

then there is a meromorphic gauge transformadion in $G \times S$ reducing $\vec{A}$ to H.

V. Analytic Galois Theory

V.i. Meromorphic Solutions and Galois Group

Consider a meromorphic automorphic system,

$$\vec{A} = \partial + \sum f_i(t)A_i, \quad f_i(t)\text{meromorphic in}S$$

we re-define,

$$S^{\times} = S \setminus \{\text{poles of } \partial \text{ and } f_i(t)\}.$$

- For a closed analytic subgroup $H \subset G$, the space of cosets G/H is an homogeneous G-space.
- Any homogeneous G-space is isomorphic to a space of cosets G/H for certain H.
- Let $\mathfrak{C}$ be the set of congugacy classes of analytic subgroups of G.
- For $\mathfrak{c} \in \mathfrak{C}$ let $M(\mathfrak{c})$ be its corresponding homogeneous space. $\vec{A}^{M(\mathfrak{c})}$ is a meromorphic Lie-Vessiot system.

V.i. Meromorphic Solutions and Galois Group (II)

- Define:

$$\mathcal{M}(\vec{A}, M(\mathfrak{c})) = \{\text{meromorphic solutions of } \vec{A}^{M(\mathfrak{c})}\}$$

- And $\mathcal{M}(\vec{A})$ the total set of meromorphic solutions:

$$\mathcal{M}(\vec{A}) = \bigcup_{\mathfrak{c}\in\mathfrak{C}} \mathcal{M}_0(\vec{A}, M(\mathfrak{c})).$$

Definition

For t_0 in $S^\times$ we define the analytic Galois group,

$$Gal_{t_0}(\vec{A}) = \bigcap_{x(t)\in\mathcal{M}(\vec{A})} H_{x(t_0)}$$

V.ii. Analytic Galois Bundle

Theorem

The Galois group $Gal_t(\vec{A})$ depends meromorphically on t in S.

Definition

We call Galois bundle of $\vec{A}$ to the complex analytic in S and meromorphic in $S^\times$ principal fiber bundle,

$$Gal(\vec{A}) = \bigcup_{t\in S^\times} Gal_t(\vec{A}) \xrightarrow{\pi} S^\times.$$

V.iv. Reduction to Galois Group

Theorem

Assume that $Gal_t(\vec{A}) \subseteq H \subseteq G$ and one of the following,

1. H is special group,
2. S is non-compact and H is connected.

Then, there is a meromorphic gauge transformation of $G \times S$ that reduces $\vec{A}$ to an automorphic system in H.

V.iv. Quadratures in Abelian Groups

- G connected abelian group, the exponential map,

$$\mathcal{R}(G) \to G, \quad A \mapsto \exp(A),$$

  is the universal covering.
- If fact it is a group morphism.
- $\mathcal{R}(G)$ is a vector group, and automorphic systems are integrated by a quadrature.
- By composition we integrate automorphic systems in G as exponential of integrals:

$$\sigma(t) = \exp\left(\sum_{i=1}^{s} \int_{t_0}^{t} f_i(\tau) A_i(\tau) d\tau\right),$$

  where $d\tau$ is the meromorphic 1-form in S dual with ∂.

V.v. Solvable Groups

Theorem

Assume that $Gal_t(\vec{A}) \subseteq H \subseteq G$, with H a connected solvable group, and one of the following,

1. H is special group
2. S is non-compact Riemann surface

then $\vec{A}$ is integrable by means of quadratures of closed meromorphic 1-forms in S and the exponential map in G.

Proof.

It is done, by Lie's reduction to H, and then by iterarion of Lie's reduction method and quadratures in quotient abelian groups.