Lie Vessiot Systems
Algebraic Dependence On Initial Conditions

David Blázquez-Sanz
david.blazquez-sanz@usa.edu.co
Juan J. Morales-Ruiz
juan.morales-ruiz@upc.edu

From Painlevé to Okamoto
June 9 - 13, 2008
The University of Tokyo, Japan

Outline

Introduction

I.i. Introduction

- A superposition law for a non autonomous differential equation
$$\frac{dx_i}{dt} = F_i(t,x) \quad i = 1, \ldots, n$$
is a set of formulae,
$$\varphi_i(x^{(1)}, \ldots, x^{(r)}, \lambda),$$
expressing the general solution,
$$x(t) = \varphi(x^{(1)}(t), \ldots, x^{(r)}(t), \lambda)$$
as function of a fundamental system of solutions, and n arbitrary constants λ_i.

I.i. Introduction (II)

- The main example is the linear superposition for solutions of a linear system,
$$\dot{x} = A(t).x$$
- n linearly independent solutions give the general solution by linear combinations.
- This is the linear superposition law for linear equations,

$$\varphi\colon \mathbb{C}^n \times \mathbb{C}^n_\lambda \to \mathbb{C}^n$$

$$\varphi(x^{(1)}, \ldots, x^{(n)}, \lambda) = \sum_{i=1}^{n} \lambda_i x^{(i)}$$

I.i. Introduction (III)

- There are non-linear equations related to linear system. They also admit superposition Laws. The main example is the Riccati equation.

$$\dot{x} = a(t) + b(t)x + c(t)x^2$$

- The anharmonic ratio of 4 different solutions is a constant of the equation,

$$\frac{d}{dt}\frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_4)(x_2 - x_4)} = 0.$$

- We can then express the fourth solution as a function of three know solutions and the constant anharmonic ratio λ,

$$x = \frac{x_3(x_1 - x_2) - \lambda x_1(x_3 - x_2)}{(x_1 - x_2) - \lambda(x_3 - x_2)}.$$

I.i. Introduction (IV)

- We can find also genuine non-linear superposition laws.
- There is also a main example, the generalized Weierstrass equation:

$$\dot{x}^2 = f(t)(x^3 - g_2 x - g_3)$$

- Its general solution is of the form,

$$x(t) = \wp\left(\int_0^t f(\tau)d\tau + \lambda\right).$$

- Weierstrass' addition formula for $\wp$ function,

$$\wp(t+u) = -\wp(t) - \wp(u) + \frac{1}{4}\left(\frac{\wp'(t) - \wp'(u)}{\wp(t) - \wp(u)}\right)^2$$

is then a superposition formula for the solutions.

I.ii. Historical Development

- Differential equations admitting superposition laws were introduced By S. Lie in 1885.
- Local conditions for the existence of a superposition law were given by S. Lie and G. Scheffers in 1893. Some other advances in the earlier theory came from E. Vessiot and A. Gulbdberg.
- In the context of mathematical physics several authors carried out the research in 20th and 21th century: Winternitz, Shnider, Shorine, Cariñena, Grabowsky, Marmo, Ramos.
- Superposition laws are also interesting for differential algebraists. The work of K. Nishioka relates superposition Laws with Kolchin's strongly normal extensions of differential fields.
- Contemporary approach to SNE, due to J. Kovacic and R. Churchill is also related with superposition laws.

Superposition and Lie-Vessiot

II.i. Algebraic Superposition Laws

Consider the following objects:

- An algebraic variety M of complex dimension n.
- A Riemann surface S provided with a meromorphic derivation ∂.
- The field K of meromorphic functions in S.
- A non-autonomous meromorphic vector field $\vec{X}$ in M with coefficiens in K,

$$\vec{X} = \partial + \sum F_i(x,t)\vec{X}_i \quad \vec{X}_i \in \mathfrak{X}(M), \quad F_i \in \mathcal{M}(M \times S)$$

- For a differential extension $K \subset L$, an L-curve in M is a point of M with coefficients in L.
- Solutions of $\vec{X}$ are L-curves tangent to $\vec{X}$.

II.i. Algebraic Superposition Laws (II)

Definition

An algebraic superposition law for $\vec{X}$ is an algebraic map,

$$\varphi\colon U \times M \to M,$$

where U is a Zariski open subset of M^r, such that, for a frame of solutions $(x^{(1)}(t), \ldots, x^{(r)}(t))$,

$$x(\lambda; t) = \varphi(x^{(1)}(t), \ldots, x^{(r)}(t), \lambda),$$

is the general solution of $\vec{X}$.

II.ii. Lie's superposition theorem

The non-autonomous vector field $\vec{X}$ can be seen as an meromorphic function $S \to \mathfrak{X}(M)$, $t_0 \mapsto \vec{X}_{t_0}$.
Define:

$$S^{\times} = S \setminus \{\text{poles of } \vec{X}\}$$

Definition (LVG Algebra)

The Lie-Vessiot-Guldberg algebra of $\vec{X}$ is the Lie subalgebra of $\mathfrak{X}(M)$ spanned by the vector fields $\vec{X}_t$ with $t \in S^{\times}$.

Theorem (S. Lie, 1893)

If $\vec{X}$ admits a superposition then its LVG algebra is finite dimensional.

The proof of this theorem hides the integration of the LVG algebra of $\vec{X}$, and some other global assumptions.

II.iii. Examples

For the linear equation $\dot{x} = A(t).x$.

- $\vec{X} = \partial + a_{ij}(t)x_j\frac{\partial}{\partial x_i}$
- The LVG algebra is contained into the algebra spanned by the fields $x_j\frac{\partial}{\partial x_i}$.
- It is a subalgebra of $gl(n,\mathbb{C})$.

For the Riccati equation, $\dot{x} = a(t) + b(t)x + c(t)x^2$.

- $\vec{X} = \partial + a(t)\frac{\partial}{\partial x} + b(t).x\frac{\partial}{\partial x} + c(t)x^2\frac{\partial}{\partial x}$
- The LVG is contained into de algebra spanned by the fields $\frac{\partial}{\partial x}, x\frac{\partial}{\partial x}, x^2\frac{\partial}{\partial x^2}$.
- It is a subalgebra of $pgl(1,\mathbb{C})$.

II.iv. Local vs Global superposition

- Lie's theorem relates a global statement with a local statement.
- To complete the picture, we need some new concepts.

Definition. Local superposition law. [CGM 2006]

A local superposition law for $\vec{X}$ is a foliation Ψ in M^{r+1}, tangent to $\vec{X}^r$, of codimension greater than n, and transversal to the fibers of the last projection $M^{r+1} \to M$.

By selecting a leave of the foliation (arbitrary constants) we can locally express the last M-coordinate as a function of the others.

II.iv. Local vs Global superposition (II)

Definition

G act algebraically pretransitively in M if there exist a positive number r and a Zarisky open subset $U \subset M^r$ such that,

$$U \simeq G \times V,$$

for certain smooth algebraic variety V.

- Analitically Transitive & Finite Rank $\Longrightarrow$ Analitically Pretransitive
- Algebraically Transitive $\Longrightarrow$ Analitically Pretransitive

- Let us consider an algebraic group G.
- Let $\mathcal{R}(G)$ be the Lie algebra of right-invariant vector fields.
- Consider an algebraic action,

$$G \times M \to M.$$

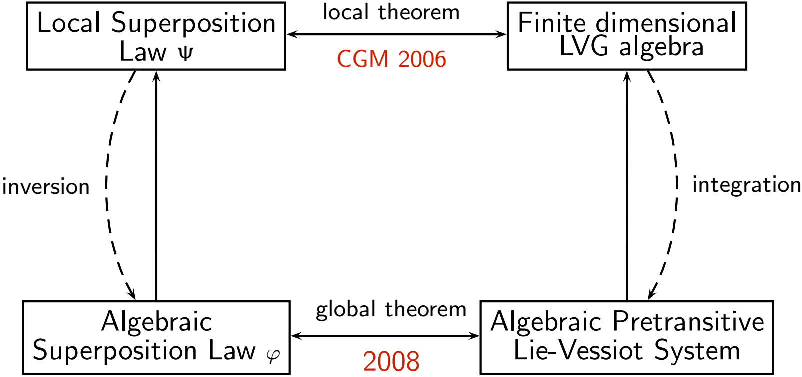
- Let $\mathcal{R}(G, M)$ the Lie algebra of fundamental fields in M.

Definition

We say that an algebraic non-autonomous vector field $\vec{X}$ in M is a Lie-Vessiot system relative to the action of G if its LVG algebra its spanned by fundamental fields;

$$\vec{X} = \partial + \sum_i f_i(t)\vec{X}_i, \quad \vec{X}_i \in \mathcal{R}(G, M).$$

II.v. Local vs Global theorem



Local Superposition Law ψ

local theorem

CGM 2006

Finite dimensional LVG algebra

inversion

integration

Algebraic Superposition Law φ

global theorem

2008

Algebraic Pretransitive Lie-Vessiot System

II.vi. Global theorem

Theorem (Lie's global superposition theorem)

$\vec{X}$ admit an algebraic superposition law if and only if it is a Lie-Vessiot system relative to an algebraically pretransitive action.

- First, let us prove that Lie-Vessiot systems admits superposition laws.
- We need some classical machinery. Vessiot's automorphic system.

II.vii. Automorphic system (I)

Consider:

- A pretransitive action of G in M.
- $\vec{X}$ a Lie-Vessiot system in M with coefficients in S.

We have an identification of the algebra of fundamental vector fields in M with the algebra $\mathcal{R}(G)$ of right-invariant vector fields in G.
Then $\vec{X}$ is transported to a right-invariant vector field in G with coefficients in S, the automorphic system.

$$\vec{A} = \partial + \sum f_i(t)\vec{A}_i, \quad f_i(t) \in K, A_i \in \mathcal{R}(G).$$

II.vii. Automorphic system (II)

- Superposition law $\Longleftrightarrow$ Group law $G \times G \to G$.
- Particular solution $\Longleftrightarrow$ General solution.
- Action $G \times M \to M \Longrightarrow$ general solution of $\vec{X}$.
- $\sigma(t)$ solution of $\vec{A}$. $\{\sigma(t) \cdot x | x \in M\}$ general solution of $\vec{X}$.
- $\{$Solutions of $\vec{A}\}$ $\Longleftrightarrow$ principal homogeneous space.

$$Sol(\vec{A}) \times G \to Sol(\vec{A})$$

$$(\sigma(t), \tau) \mapsto \sigma(t) \cdot \tau$$

II.viii. Lie-Vessiot $\Longrightarrow$ superposition

Consider a Lie-Vessiot system $\vec{X}$ in M.

- We have a pretransitive action of G.
- There is a $U \subset M^r$ such that $U \simeq G \times V$.
- Consider the natural projection $\pi \colon U \to G$.
- Define,

$$\varphi \colon W \times M \to M, \quad (\bar{x}, y) \to s(\bar{x}) \cdot y$$

$$\psi \colon W \times M \to M, \quad (\bar{x}, y) \to s(\bar{x})^{-1} \cdot y$$

Lemma

φ is a superposition law. Fibers of ψ define a local superposition law.

II.ix. Superposition $\Longrightarrow$ Lie-Vessiot (I)

Assume that $\vec{X}$ admits a superposition law φ. Define,

$$\varphi\colon U \times M^r \to M^r, \quad (\bar{x}, \bar{y}) \mapsto (\varphi(\bar{x}, y_i))$$

it is a superposition law for $\vec{X}^r$ in M^r.

Lemma

Without loss of generality we can assume that for all $\bar{x}, \bar{y}$ in U there exist $y \in M^r$ such that $\bar{\varphi}(\bar{z}, \bar{y}) = \bar{z}$.

For $\bar{x} \in U$, define $\sigma_{\bar{x}}\colon M \to M$, $y \mapsto \varphi(\bar{x}, y)$.

$$G = \{\sigma_{\bar{x}} | \bar{x} \in U\}.$$

Lemma $\Longrightarrow$ G is a group of automorphisms.

II.ix. Superposition $\Longrightarrow$ Lie-Vessiot (II)

Lemma

G is a complex analytic Lie group.

Sketch of proof:

- Tangent vectors in U span vector fields in M through φ.
- The tangent space a point of U spans a Lie algebra of vector fields in M.
- The tangent space to G is identified with this Lie algebra.
- Fibers of $\pi\colon U \to G$ are closed sub-manifolds of constant dimension.

II.ix. Superposition $\Longrightarrow$ Lie-Vessiot (III)

Lemma

The G acts algebraically pretransitively.

Sketch of proof.

- First we prove that the action of G is analitically pretransitive.
- We have $U = G \times U_0$, with U_0 algebraic.
- There are natural sections of G into U that gives us the algebraic structure of G.
- These sections are compatible with the action: G is algebraic and acts algebraically pretransitively in M.

Galois theory

III.i. Schemes with derivation

Consider:

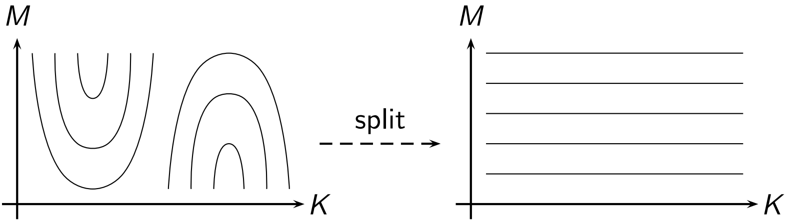
- A ch 0, algebraically closed field C.
- A differential field K, with derivation ∂, and constant field C.
- A C-scheme M.
- The extended phase space, $M \times_C Spec(K) = M_K$.

Definition

A non-automonous vector field in M with coefficients in K is a derivation $\vec{X}\colon \mathcal{O}_{M_K} \to \mathcal{O}_{M_K}$ compatible with ∂ in the sense $\vec{X} f(t) = \partial f(t)$, $\forall f(t) \in K$.

III.ii. Split of schemes with derivation

- The pair $(M_K, \vec{X})$ is a scheme with a derivation.
- Schemes with derivations for a category embracing the category of schemes.
- $M \times_C (Spec(K), \partial)$ is a scheme with derivation. We say that this scheme is dynamically trivial in M.
- We say that $(M_K, \vec{X})$ splits if there is a K-isomorphism (splitting morphism)$(M_K, \vec{X}) \to M \times_C (Spec(K), \partial)$.



III.iii. Notion of solution curve

- Certain points of M_K are invariant for derivation $\vec{X}$.
- They form a subspace $Diff(M_K, \vec{X})$ that inherits the topological structure M_K.
- M_K is a Keigher scheme $\Longrightarrow$ $Diff(M_K, \vec{X})$ is a differential scheme.
- Closed points $\mathfrak{x} \in Diff(M_K, \vec{X})$ define solution curves of the vector field $\vec{X}$.
- They are solutions with coefficients in the rational field of the corresponding closed point.

$$(Spec(\kappa(\mathfrak{x})), \partial) \longrightarrow (M_K, \vec{X})$$

III.iv. Automorphic system

Consider:

- An algebraic group G over C.
- A G-homogeneous space M, with a faithful action.
- $\vec{X}$ a Lie-Vessiot system in M with coefficients in K. In particular it is a non-autonomous vector field in M.

We have an identification of the algebra of fundamental vector fields in M with the algebra $\mathcal{R}(G)$ of right-invariant vector fields in G.
Then $\vec{X}$ is transported to a right-invariant vector field in G with coefficients in K, the automorphic system

$$\vec{A} = \partial + \sum f_i(t)\vec{A}_i, \quad f_i(t) \in K, A_i \in \mathcal{R}(G).$$

Finally we have a hierarchy of vector fields in the family of G-spaces.

III.v. Geometry of the automorphic system

- For the automorphic system, the superposition formula is the group law $G \times_C G \to G$.
- One particular solution gives the general solution (split). Assume that $\mathit{Diff}(G_K, \vec{A})$ has a rational point σ. Then the left translation $L_{\sigma^{-1}}$ is a splitting isomorphism,

$$L_{\sigma^{-1}} \colon (G_K, \vec{A}) \xrightarrow{\sim} G \times_C (\mathit{Spec}(K), \partial).$$

- Moreover, the action $G \times M \to M$, gives us also the general solution of $\vec{X}$, the Lie-Vessiot system in M.
- The action of G in M is faithful $\Longrightarrow$ $(M_K, \vec{X})$ splits if and only if $(G_K, \vec{A})$ split.

III.vi. Splitting field

- Let L be the field of a closed point $\mathfrak{x} \in Diff(G_K, A)$.
- In such case $(G_L, \vec{A})$ has a rational point, and then splits.

$$\begin{tikzcd}
Diff(G_K, \vec{A}) \times_C G \arrow[r] & Diff(G_K, \vec{A}) \\
Diff(G_L, \vec{A}) \times_C G \arrow[r] \arrow[u] & Diff(G_L, \vec{A}) \arrow[u]
\end{tikzcd}$$

Results.

- $K \subset L$ is a strongly normal extension $\longrightarrow$ Galois correspondence.
- Consider a differential extension $K \subset S$. If $(G_S, \vec{A})$ splits then $L \subset S$.
- L does not depend on $\mathfrak{x}$.

III.vii. Galois Group

Consider a closed point $\mathfrak{x} \in Diff(G_K, \vec{A})$ and the action,

$$Diff(G_K, \vec{A}) \times_C G \to G$$

Definition

We call Galois group of $\vec{A}$ at $\mathfrak{x}$, to the isotropy subgroup of $\mathfrak{x}$, $H_{\mathfrak{x}}$.

Theorem

The Galois group is an algebraic subgroup of G. It is canonically isomorphic to the group of differential K-automorphisms of L. All Galois groups for $(G_K, \vec{A})$ are conjugated.

Lie-Kolchin reduction

IV.i. Lie's reduction method

How to integrate Lie-Vessiot equations. Theoretical tools:

Logarithmic derivative

$l\partial\colon G(K) \to K \otimes_C \mathcal{R}(G), \quad \sigma \mapsto R'_{\sigma^{-1}}(\sigma').$

Gauge transform

Let σ a K-rational point. Then the L_σ transforms $\vec{A}$ onto $Adj_\sigma(\vec{A}) + l\partial(\sigma)$

Automorphic equation

σ is a solution of $\vec{A}$ if and only if $l\partial(\sigma) = \vec{A}$.

IV.i. Lie's reduction method (II)

Consider a G-homogeneous space N and $\vec{X}^N$ the Lie-Vessiot system induced by $\vec{A}$.

Lemma

If there is a constant solution $x_0 \in N(C)$ of $\vec{X}^N$ then $\vec{A} \in \mathcal{R}(H_{x_0}) \otimes_C K \subset \mathcal{R}(G) \otimes_C K$.

Assume that there is a K-solution, x, of $\vec{X}^N$. If there exist $\sigma \in G(K)$ such that $\sigma x_0 = x$ is constant, then $L_{\sigma^{-1}} \colon \vec{A} \mapsto \vec{B} \in \mathcal{R}(H_{x_0}) \otimes_C K$.
We shall study the fibers of the quotient $\pi \colon G_K \to N_K$. The isotropy group H_x acts freely and transitively in the fiber of x. Then $\pi^{-1}(x)$ is a principal H_x-homogeneous space. And $H_x \simeq H_{x_0} \times_C Spec(K)$.

IV.ii. Lie-Kolchin reduction method

Principal homogeneous spaces are classified by Galois cohomology. In particular if $H^1(H_{x_0}, K) = 0$ then al $(H_{x_0})_K$ homogeneous space is isomorphic to $(H_{x_0})_K$ and then has a rational point.

Theorem. Lie-Kolchin reduction method

Assume that $\vec{X}^N$ has a K-solution, and the isotropy group of any point x_0 in $N(C)$ has trivial Galois cohomology $H^1(H_{x_0}, K) = 0$. There is a gauge transform,

$$\vec{A} \to \vec{B} \in \mathcal{R}(H_{x_0}) \otimes_C K.$$

reducing $\vec{A}$ to the Lie algebra of the isotropy group.

IV.iii. Kolchin's reduction theorem

Kolchin's reduction theorem

Let H be the Galois group of $(G_K, \vec{A})$. Let $\bar{H} \subset G$ be a group including H, and such that the quotient $G/\bar{H}$ exist. There is an algebraic extension $K \to \tilde{K}$ and gauge transformation of $G_{\tilde{K}}$ such that

$$\vec{A} \mapsto \vec{B} \in \mathcal{R}(H) \otimes_C \tilde{K}$$

Sketch of proof. We have $\mathfrak{x} \in G_K$, and $H = H_{\mathfrak{x}}$. Consider σ a solution of $\vec{A}$ such that $\sigma \mapsto \mathfrak{x}$. Consider the projection $\pi \colon G_L \to (G/H)_L = N_L$. $\pi(\sigma) = x$ is a L-point of N_L, and a solution of $\vec{X}^N$. K-automorphisms of L maps x to x. Then x is a K-point of N.

IV.iv. Liouville's theorem

Consider $B \subset G$ a maximal connected solvable subgroup (Borel subgroup). We call flag-variety of G to the homogeneous space $F = G/B$

$$\begin{aligned} GL(2,C) &\longrightarrow \mathbb{P}(1,C) \\ SL(2,C) &\longrightarrow \mathbb{P}(1,C) \\ SO(3,C) &\longrightarrow \mathbb{P}(1,C) \\ GL(3,C) &\longrightarrow \mathbb{P}(T\mathbb{P}(2,C)) \end{aligned}$$

Theorem. Liouville

Consider $\vec{F}$ the Lie-Vessiot system induced by $\vec{A}$ in F. The connected component of the identity of the Galois group is solvable if and only if the $\vec{F}$ has an algebraic solution.

IV.v. Applications

Corollary. Liouville Theorem.

The linear equation

$$y'' + p(t)y' + q(t) = 0$$

has Liouvillian solutions if and only if the Riccati equation

$$\dot{x} = q(t) + p(t)x + x^2$$

has an algebraic solution.

IV.v. Applications (II)

Theorem

The linear system

$$\begin{pmatrix} \dot{x}_0 \\ \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} & a & b \\ -a & & c \\ -b & -c & \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \qquad a, b, c \in K.$$

has Liouvillian solutions if and only if the Riccati equation

$$\dot{x} = \frac{-b - ic}{2} - iax + \frac{-b + ic}{2} x^2$$

has an algebraic solution.

IV.v. Applications (III)

Theorem

The Picard-Vessiot extension of the equation,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

is Liouvillian, if and only if the system,

$$\left\{ \begin{array}{rcl} \dot{x} & = & a_{21} + (a_{22} - a_{11})x + a_{23}y - a_{12}x^2 - a_{13}xy \\ \dot{y} & = & a_{31} + a_{32}x + (a_{33} - a_{11})y - a_{12}xy - a_{13}y^2 \end{array} \right.$$

$$\dot{z} = a_{32} - a_{12}y + (a_{33} - a_{22} + a_{12}y - a_{13}y)z + (a_{13}y - a_{23})z^2$$

has an algebraic solution.

IV.vi. A note on the role of symmetries

- Consider a linear system: $\dot{U} = AU$. The corresponding vector field is $\vec{A} = \partial + a_{ij}(t)x_j\frac{\partial}{\partial x_i}$
- Linear symmetries of the system: Vector fields of the form $\vec{L} = l_{ij}(t)x_j\frac{\partial}{\partial x_i}$ such that $[\vec{L}, \vec{A}] = 0$.
- $Lie(\vec{A}, K)$, the set of linear symmetries of $\vec{A}$, is a finite dimensional Lie algebra. If we specialize to a value t_0 of t, it is represented as a subalgebra $Lie_{t_0}(A, C(t)) \subset \mathcal{R}(G)$.

Theorem

The connected component of the identity of the Galois group is isomorphic to a subgroup of the centralizer group of $Lie_{t_0}(A, K)$,

$$Z = \{U | UL = LU \ \forall L \in Lie_{t_0}(\vec{A}, C(t))\}.$$

IV.vi. A note on the role of symmetries (II)

Corollary. Linear Morales-Ramis.

Consider $\vec{X}$, a n-degreees of freedom non-autonomous linear Hamiltonian system generated by a quadratic homogeneous Hamiltonian function $H(x_i, y_i) \in K[x_i, y_i]$; and let $\vec{A}$ be its associated automorphic system. If there are n independent rational invariants $F_1(x,y), \ldots, F_n(x,y) \in K(x_i, y_i)$ such that $\{F_i, F_j\} = 0$, then the Galois group of $\vec{A}$ is virtually abelian.

References

S. Lie.
Vorlesungen über continuerliche Gruppen.
Teubner, Leizpig, 1893.

E. Kolchin
Differential Algebra and Algebraic Groups.
Academic Press, New York 1973.

E. Vessiot.
Sur les systèmes d'équations différentielles du premier ordre qui ont des systèmes fondamentaux d'intégrales.
Annales de la faculté des sciences de Toulouse, Sr. 1, 8 no. 3 (1894), p. H1–H33.