

Global & Local Properties of Lie-Vessiot Systems

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Outline

- I. Motivation
- II. Lie's Superposition Theorem
- III. Lie-Vessiot Hierarchy and Galois Theory
- IV. Lie's Reduction Method

I. Motivation

I.i. Motivation

- ▶ A **superposition law** for a non autonomous differential equation

$$\frac{dx_i}{dt} = F_i(t, x) \quad i = 1, \dots, n$$

is a set of formulae,

$$\varphi_i(x^{(1)}, \dots, x^{(r)}, \lambda),$$

expressing the general solution,

$$x(t) = \varphi(x^{(1)}(t), \dots, x^{(r)}(t), \lambda)$$

as function of a **fundamental system of solutions**, and n arbitrary constants λ_i .

I.i. Motivation (II)

- ▶ The main example is the linear superposition for solutions of a linear system,

$$\dot{x} = A(t).x$$

- ▶ n linearly independent solutions give the general solution by linear combinations.
- ▶ This is the linear superposition law for linear equations,

$$\varphi: \mathbb{C}^n \times \mathbb{C}_\lambda^n \rightarrow \mathbb{C}^n$$

$$\varphi(x^{(1)}, \dots, x^{(n)}, \lambda) = \sum_{i=1}^n \lambda_i x^{(i)}$$

I.i. Motivation (III)

- ▶ There are non-linear equations related to linear system. They also admit superposition Laws. The main example is the **Riccati equation**.

$$\dot{x} = a(t) + b(t)x + c(t)x^2$$

- ▶ The anharmonic ratio of 4 different solutions is a constant of the equation,

$$\frac{d}{dt} \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_4)(x_2 - x_3)} = 0.$$

- ▶ We can then express the fourth solution as a function of three know solutions and the constant anharmonic ratio λ ,

$$x = \frac{x_3(x_1 - x_2) - \lambda x_1(x_3 - x_2)}{(x_1 - x_2) - \lambda(x_3 - x_2)}.$$

I.i. Motivation (IV)

- ▶ We can find also genuine non-linear superposition laws.
- ▶ There is also a main example, the generalized **Weierstrass equation**:

$$\dot{x}^2 = f(t)(x^3 - g_2x - g_3)$$

- ▶ Its general solution is of the form,

$$x(t) = \wp \left(\int_0^t f(\tau) d\tau + \lambda \right).$$

- ▶ Weierstrass' addition formula for \wp function,

$$\wp(t+u) = -\wp(t) - \wp(u) + \frac{1}{4} \left(\frac{\wp'(t) - \wp'(u)}{\wp(t) - \wp(u)} \right)^2$$

is then a superposition formula for the solutions.

I.ii. Historical Development

- ▶ Differential equations admitting superposition laws were introduced By S. Lie in 1885.
- ▶ Local conditions for the existence of a superposition law were given by S. Lie and G. Scheffers in 1893. Some other advances in the earlier theory came from E. Vessiot and A. Gulbberg.
- ▶ In the context of mathematical physics several authors carried out the research in 20th and 21th century: Winternitz, Shnider, Shore, Cariñena, Grabowsky, Marmo, Ramos.
- ▶ Superposition laws are also interesting for differential algebraists. The work of K. Nishioka relates superposition Laws with Kolchin's strongly normal extensions of differential fields.
- ▶ Contemporary approach to SNE, due to J. Kovacic and R. Churchill is also related with superposition laws.

II. Lie's Superposition Theorem

II.i. Superposition Laws

Let \vec{X} be a non-autonomous vector field in M with time varying in a Riemann surface S , with a fixed meromorphic derivation ∂ :

$$\vec{X} = \partial + \sum_{i=1}^n f_i(x, t) \frac{\partial}{\partial x_i}.$$

Definition

A superposition law for \vec{X} is an analytic map,

$$\varphi: U \times M \rightarrow M,$$

where:

- ▶ U is an analytic subset of M^r
- ▶ U is union of orbits of \vec{X}^r .

Such that, for any local solution $(x^{(1)}(t), \dots, x^{(r)}(t))$ of \vec{X}^r taking values in U , defined for t varying in $S' \subset S$,

$$x_\lambda(t) = \varphi(x^{(1)}(t), \dots, x^{(r)}(t), \lambda)$$

II.ii. Lie's Superposition Theorem

- ▶ Consider $\vec{X} = \partial + \sum_i f_i(x, t) \frac{\partial}{\partial x_i}$, meromorphic in $S \times M$ and holomorphic in $S^\times \times M$.
- ▶ For $t_0 \in S^\times$, define \vec{X}_{t_0} in $\mathfrak{X}(M)$,

$$\vec{X}_{t_0} = \sum f_i(x, t_0) \frac{\partial}{\partial x_i}.$$

- ▶ \vec{X} is seen as a map $\vec{X}: S^\times \rightarrow \mathfrak{X}(M)$

Definition

We call Lie-Vessiot-Guldberg algebra of \vec{X} to the Lie algebra $\mathfrak{g}(\vec{X})$ of vector fields in M spanned by the image of the above defined map.

Theorem (Lie-Scheffers 1893)

A necessary condition for the existence of a superposition law for \vec{X} is $\dim \mathfrak{g}(X) < \infty$.

II.iii. Pretransitive Lie Group Actions

- ▶ Consider a complex analytic group G acting in M ,

$$G \times M \rightarrow M, \quad (\sigma, x) \mapsto \sigma \cdot x.$$

- ▶ $\mathcal{R}(G)$ right invariant fields in G .
- ▶ $\mathcal{R}(G, M)$ fundamental fields in M .
- ▶ We say that G acts **pretransitively** in M if there is a G -invariant analytic open subset $U \in M^r$ such that the quotient, $U \rightarrow U/G$ is a principal bundle.

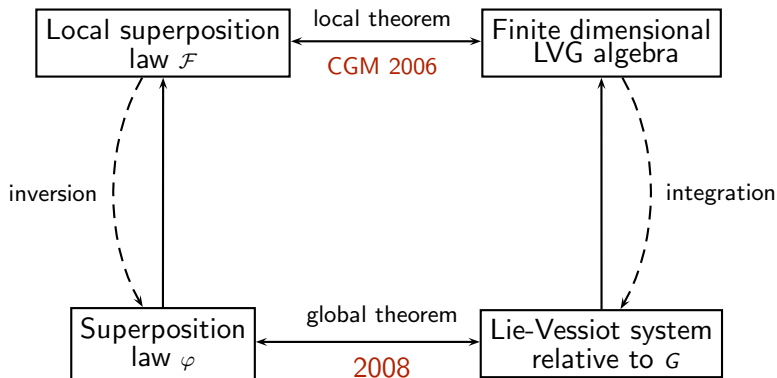
Definition

A non-autonomous vector field \vec{X} is called a Lie-Vessiot system associated to the action of G in M if $\mathfrak{g}(\vec{X}) \subset \mathcal{R}(G, M)$.

Theorem (2008)

A non-autonomous vector field \vec{X} in M admit a superposition law if and only if it is a Lie-Vessiot system associated to a pretransitive action in M .

II.iv. Lie's Superposition Theorem (Scheme)



III. Lie-Vessiot Hierarchy and Galois

III.i. Automorphic System (I)

Definition

We call **automorphic system** to any Lie-Vessiot system in a Lie group G .

- ▶ The superposition law for an automorphic system is the composition law,

$$G \times G \rightarrow G.$$

- ▶ For small $S' \subset S$ the space of solutions $\mathcal{G}_{\bar{A}}$ is a principal homogeneous space by the right side,

$$\mathcal{G}_{\bar{A}} \times G \rightarrow G, \quad (\sigma(t), \tau) \mapsto \sigma(t) \cdot \tau$$

- ▶ **Example:** Let $U(t)$ be a fundamental matrix of solutions of $\dot{x} = A(t).x$. For each non degenerate constant matrix $C \in GL(n, \mathbb{C})$, $U(t).C = V(t)$ is another fundamental matrix of solutions.

III.i. Automorphic System (II)

Let M be a faithful G -homogeneous space of finite rank r .

- ▶ Let \vec{X} be a Lie-Vessiot system in M with t varying in S . Then, there are functions $f_i(t)$ in S^\times with:

$$\vec{X} = \sum_{i=1}^s f_i(t) A_i^M, \quad \mathcal{R}(G, M) = \langle A_1^M, \dots, A_s^M \rangle.$$

- ▶ By the isomorphism $\mathcal{R}(G, M) \simeq \mathcal{R}(G)$ we define the associated automorphic system,

$$\vec{A} = \sum_{i=1}^s f_i(t) A_i.$$

- ▶ **Example:** $\dot{x} = A(t)x \rightsquigarrow \dot{U} = A(t)U.$

III.ii. Lie-Vessiot Hierarchy

- ▶ G -space morphisms,

$$\phi: M \rightarrow N, \quad \phi(\sigma \cdot x) = \sigma \cdot \phi(x),$$

carry Lie-Vessiot systems onto Lie-Vessiot systems.

- ▶ Let \vec{A} be an automorphic system in G . Throughout the morphism,

$$\mathcal{R}(G) \rightarrow \mathcal{R}(G, M),$$

it induces a Lie-Vessiot system \vec{A}^M in any G -space M .

- ▶ If M is a faithful G -homogeneous G -space, then \vec{A} is the automorphic system associated to \vec{A}^M .
- ▶ Any G -space morphism $M \rightarrow N$ carry \vec{A}^M onto \vec{A}^N .

III.iii. Meromorphic Solutions and Galois Group

Consider a **meromorphic** automorphic system,

$$\vec{A} = \partial + \sum f_i(t)A_i, \quad f_i(t) \text{ meromorphic in } S$$

we re-define,

$$S^\times = S \setminus \{\text{poles of } \partial \text{ and } f_i(t)\}.$$

- ▶ For a closed analytic subgroup $H \subset G$, the space of cosets G/H is an homogeneous G -space.
- ▶ Any homogeneous G -space is isomorphic to a space of cosets G/H for certain H .
- ▶ Let \mathfrak{C} be the set of conjugacy classes of analytic subgroups of G .
- ▶ For $\mathfrak{c} \in \mathfrak{C}$ let $M(\mathfrak{c})$ be its corresponding homogeneous space. $\vec{A}^{M(\mathfrak{c})}$ is a meromorphic Lie-Vessiot system.

III.iii. Meromorphic Solutions and Galois Group (II)

- ▶ Define:

$$\mathcal{M}(\vec{A}, M(c)) = \{\text{meromorphic solutions of } \vec{A}^{M(c)}\}$$

- ▶ And $\mathcal{M}(\vec{A})$ the total set of meromorphic solutions:

$$\mathcal{M}(\vec{A}) = \bigcup_{c \in \mathcal{C}} \mathcal{M}(\vec{A}, M(c)).$$

Definition

For t_0 in S^\times we define the analytic Galois group,

$$\text{Gal}_{t_0}(\vec{A}) = \bigcap_{x(t) \in \mathcal{M}(\vec{A})} H_{x(t_0)}$$

III.iv. Analytic Galois Bundle

Theorem

The Galois group $Gal_t(\vec{A})$ depends meromorphically on t in S .

Definition

We call Galois bundle of \vec{A} to the complex analytic in S and meromorphic in S^\times principal fiber bundle,

$$Gal(\vec{A}) = \bigcup_{t \in S^\times} Gal_t(\vec{A}) \xrightarrow{\pi} S^\times.$$

IV. Lie's Reduction Method

IV.i. Gauge Transformations

- ▶ Consider $S' \subset S$ and a curve $\sigma(t) \in \mathcal{O}(S', G)$. We define the **gauge transformations**,

$$L_{\sigma(t)}: G \times S' \rightarrow G \times S', \quad (\tau, t) \mapsto (\sigma(t) \cdot \tau, t).$$

$$L_{\sigma(t)}: M \times S' \rightarrow M \times S', \quad (x, t) \mapsto (\sigma(t) \cdot x, t).$$

- ▶ They transform automorphic (Lie-Vessiot) systems \vec{A} onto automorphic (Lie-Vessiot) systems.

IV.ii. Lie's theorem on Reduction

For $H \subset G$, we have $\mathcal{R}(H) \subset \mathcal{R}(G)$. Automorphic systems in H are, in particular, automorphic systems in G . Let us consider:

- ▶ \vec{A} automorphic in G .
- ▶ M homogeneous G -space.
- ▶ \vec{X} induced Lie-Vessiot in M .
- ▶ $x_0 \in M$, and $H \subset M$ its isotropy group.

Lemma

If $x(t) = x_0$ is a solution of \vec{X} , then \vec{A} is an automorphic system in $H \subset G$.

IV.ii. Lie's theorem on Reduction (II)

- ▶ Let $x(t)$ be a particular solution of \vec{X} .
- ▶ Assume that there exist a curve $\sigma(t)$, such that $\sigma(t) \cdot x_0 = x(t)$.
- ▶ The Gauge transformation $L_{\sigma(t)^{-1}}$ sends \vec{A} to a new automorphic system \vec{B} .
- ▶ The lemma applies to \vec{B} .

The problem is to find a Global $\sigma(t)$!

IV.ii. Lie's theorem on Reduction (III)

For a given meromorphic $x: S \rightarrow M$ we consider.

$$\mathcal{H} = \{(\sigma, t) \mid \sigma \cdot x_0 = x(t)\} \subset G \times S,$$

projection $\pi: \mathcal{H} \rightarrow S$ is a fiber bundle. Our gauge transformation comes from a global section.

- ▶ If S is non-compact and H is connected, **Grauert theorem** ensures that there is a global meromorphic section.
- ▶ We say that H is **special** if any principal bundle modeled over H has a global meromorphic section.

Theorem

Assume that there is a meromorphic solution $x(t)$ of \vec{X} defined in S . Assume one of the following:

1. H is special
2. S is non-compact and H connected

then there is a meromorphic gauge transformation in $G \times S$ reducing \vec{A} to H .

IV.iii. Reduction to Galois Group

Theorem

Assume that $Gal_t(\vec{A}) \subseteq H \subseteq G$ and one of the following,

1. H is special group,
2. S is non-compact and H is connected.

Then, there is a meromorphic gauge transformation of $G \times S$ that reduces \vec{A} to an automorphic system in H .

IV.iv. Quadratures in Abelian Groups

- ▶ G connected abelian group, the exponential map,

$$\mathcal{R}(G) \rightarrow G, \quad A \mapsto \exp(A),$$

is the universal covering.

- ▶ In fact it is a group morphism.
- ▶ $\mathcal{R}(G)$ is a vector group, and automorphic systems are integrated by a quadrature.
- ▶ By composition we integrate automorphic systems in G as exponential of integrals:

$$\sigma(t) = \exp \left(\sum_{i=1}^s \int_{t_0}^t f_i(\tau) A_i(\tau) d\tau \right),$$

where $d\tau$ is the meromorphic 1-form in S dual with ∂ .

IV.v. Solvable Groups

Theorem

Assume that $Gal_t(\vec{A}) \subseteq H \subseteq G$, with H a connected solvable group, and one of the following,

1. H is special group
2. S is non-compact Riemann surface

then \vec{A} is integrable by means of quadratures of closed meromorphic 1-forms in S and the exponential map in G .

Proof.

It is done, by Lie's reduction to H , and then by iteration of Lie's reduction method and quadratures in quotient abelian groups.

IV.vi. Symmetries and Reduction

- ▶ Infinitesimal symmetries of \vec{A} define a sheaf of vector fields in $M \times S$.
- ▶ Any symmetry reduces to its transversal part,

$$\vec{L} = \sum_i f(\sigma, t) \frac{\partial}{\partial \sigma_i}, \quad [\vec{L}, \vec{A}] = 0.$$

- ▶ The sheaf of transversal symmetries is characterized as follows:

$$\text{Lie}(\vec{X}) \simeq \mathcal{L}(G) \otimes_{\mathbb{C}} \mathcal{O}_{S \times G}^{\vec{A}},$$

- ▶ Where $L(G)$ is the algebra of left-invariant vector fields in G , and $\mathcal{O}_{S \times G}^{\vec{A}}$ is the sheaf of first integrals of \vec{A} .

IV.vi. Symmetries and Reduction (II)

- ▶ A General symmetries of \vec{A} does not descend to a symmetrie the Lie-Vessiot system \vec{X} .
- ▶ Descending symmetries are right-invariant symmetries,

$$L = \sum_i f(t)\vec{A}_i, \quad [\mathcal{L}, \vec{A}] = 0, \quad \vec{A}_i \in \mathcal{R}(G).$$

- ▶ The right invariant symmetries are determined by a **Lie-Vessiot** system, the system induced in $\mathcal{R}(G)$ by the adjoint action of G .

$$\partial \vec{L} = [\vec{A}, \vec{L}]$$

- ▶ We can apply Lie's reduction by terms of known symmetries.

IV.vi. Symmetries and Reduction (III)

- ▶ Let $Right(\vec{A})$ be the space of meromorphic right invariant symmetries of \vec{A} .
- ▶ For $t \in S^\times$, $Right_t(\vec{A})$ is a Lie subalgebra of $\mathcal{R}(G)$.

Theorem

For all t in S^\times , The centralizer subgroup of $Right_t(\vec{A})$ contains the Galois group $Gal_t(\vec{A})$.

Corollary.

Let \vec{A} be an automorphic system of the symplectic group $Sp(2n, C)$. If $Right(\vec{A})$ contains an abelian algebra of dimension n , then the Galois group of \vec{A} is virtually abelian.